REMARK ON A METHOD OF CONSTRUCTING LIAPUNOV FUNCTIONS

(ZAMECHANIE OB ODNOM SPOSOBE POSTROENIIA FUNKTSII LIAPUNOVA)

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We shall consider nonlinear control systems described by the equations

$$\dot{x}_k = \sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + h_k f(\sigma), \qquad \sigma = \sum_{s=1}^{n} j_s x_s \qquad (k = 1, \ldots, n) \tag{1}$$

where $b_{k\alpha}$, h_k, j_s are constants and $f(\sigma)$ is a nonlinear function determined to within the relationship

$$\sigma f(\sigma) > 0 \tag{2}$$

We shall seek a Liapunov function for the system (1) of the form

$$V = \Phi + \beta \int_0^\sigma f(\sigma)\, d\sigma, \qquad \Phi = \frac{1}{2} \sum_{k=1}^{n} \sum_{\alpha=1}^{n} m_{k\alpha} x_k x_\alpha \tag{3}$$

Here Φ is a quadratic form in the variables $x_1, \ldots, x_n$ of constant sign (or definite sign), and β is a constant such that

$$\operatorname{sign} \beta = \operatorname{sign} \Phi \tag{4}$$

The stability of the zero solution of system (1) was investigated with the aid of Liapunov functions of the form (3) in the series of papers [1,2,3]. In paper [4]* the following method for establishing criteria of stability by means of Liapunov functions of the form (3) was proposed, which was used also in certain subsequent papers.

The derivative of function (3), by virtue of system (1), takes the form

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* The exposition of paper [4] is also contained in [2] and [3].

$$\dot{V} = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{\alpha=1}^{n} m_{ki} b_{k\alpha} x_i x_\alpha + f(\sigma) \sum_{i=1}^{n} \sum_{k=1}^{n} m_{ki} h_k x_i + \\ + \beta f(\sigma) \sum_{s=1}^{n} \sum_{\alpha=1}^{n} b_{s\alpha} j_s x_\alpha + \beta f^2(\sigma) \sum_{s=1}^{n} j_s h_s \tag{5}$$

and can be considered as a quadratic form in the $n+1$ variables $x_1, \ldots, x_n, f(\sigma)$.

In paper [4] the conditions of stability for system (1) were formulated as conditions for the existence of $m_{k\alpha}$ and β such that function (3) was of definite sign and function (5) was a quadratic form in the variables $x_1, \ldots, x_n, f(\sigma)$ of definite sign opposite to that of V.

It is found, however, that such a method of constructing a Liapunov function is not effective. Specifically, the following assertion is true: it is impossible to select the variables of system (1) such that there exists a function of the form (3), of definite sign for any function (2), and such that its derivative by virtue of system (1) is a quadratic form in the $n+1$ variables $x_1, \ldots, x_n, f(\sigma)$ having definite sign opposite to that of V.

To prove this we assume the opposite, namely that the function V mentioned in the formulation of the theorem exists. For definiteness we shall assume that $V > 0$.

Then the quadratic form

$$V_k = \Phi + \beta \frac{c\sigma^2}{2} \tag{6}$$

turns out to be strictly positive for all $c > 0$, and its derivative by virtue of the linear system

$$\dot{x}_k = \sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + h_k f(\sigma), \qquad \sigma = \sum_{s=1}^{n} j_s x_s \qquad (k = 1, \ldots, n) \tag{7}$$

resulting from (5) on replacing $f(\sigma)$ by $c\sigma$, is a quadratic form, strictly negative for all c ($-\infty < c < \infty$).

By virtue of the assumption, the characteristic equation of system (7)

$$D(\lambda) - cM(\lambda) = 0 \tag{8}$$

(where D and M are polynomials [1]) has, for all $c > 0$, roots with negative real parts.

We assume first that $M(0) \neq 0$. We shall decrease c and let $c^* < 0$ be the largest number for which the equation

$$D(\lambda) - c^* M(\lambda) = 0 \tag{9}$$

has a root with real part equal to zero. Such a number always exists since $M(0) \neq 0$. Then by virtue of the assumption and the known theorem of Liapunov we obtain that function (6) is strictly positive for $c > c^*$.

From this it follows that for $c = c^*$ two possibilities arise:

1. Form (6) is strictly positive.

2. Form (6) has constant positive sign.

But if the first case holds then because of Liapunov's theorem it would result that the zero solution of system (7) for $c = c^*$ is asymptotically stable, which contradicts the choice of the number c^*.

The second possibility, as was shown by Malkin [5], in general cannot be realized. The contradiction obtained proves the theorem for the case $M(0) \neq 0$. However, the proof carries over easily also to the case $M(0) = 0$, if one takes into account that the realization of $M(0) \neq 0$ can be attained by as small a change of the coefficients as desired.

It is possible to give also a purely algebraic proof for the theorem just proved. For this we remark that the expression (5) may be rewritten in the form

$$\dot{V} = \sum_{k=1}^{n} \Big(\sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha + h_k y \Big) \Big(\sum_{s=1}^{n} m_{ks} x_s + j_k y \Big) \qquad (y = f(\sigma)) \tag{10}$$

If

$$D(0) = \det | b_{k\alpha} | \neq 0 \tag{11}$$

then given an arbitrary value $y = y_0 \neq 0$, we determine the quantities x_i° $(i = 1, 2, \ldots, n)$ from the relations

$$\sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha^\circ = - h_k y_0 \qquad (k = 1, \ldots, n) \tag{12}$$

Among the x_α° there are definitely numbers different from zero, and at the same time we have

$$\dot{V}(x_1^\circ, \ldots, x_n^\circ, y_0) = 0$$

that is, for condition (11) the derivative $\dot{V}$ does not have definite sign. But if $D(0) = 0$ then it is possible to select $y_0 = 0$ and the quantities x_i° as a non-zero solution of the system

$$\sum_{\alpha=1}^{n} b_{k\alpha} x_\alpha{}^\circ = 0 \qquad (k = 1, \ldots, n)$$

since in this connection again $\dot{V}(x_1{}^\circ, \ldots, x_n{}^\circ, y_0) = 0$, and then, even for $D(0) = 0$, $\dot{V}$ does not turn out to be of definite sign in the $n + 1$ arguments. Q. E. D.

At the same time it is necessary to note that there exist functions of the form (3) with definite sign, the derivatives of which may be functions of $n + 1$ variables with fixed sign. In particular, sufficient conditions for the existence of such a function are given by solvability conditions for quadratic equations in [6].

In the general case the derivative of a Liapunov function of the form (3) may be a function with definite sign in the variables $x_1, \ldots, x_n$ and with variable sign if it is considered as a function of $n + 1$ arguments [7].

It is possible to avoid the difficulty discussed above by requiring definiteness of sign only of the quadratic form obtained from (5) by replacing

$$f(\sigma) \text{ by } c\left(\sum_{s=1}^{n} j_s x_s\right)$$

In this connection it is possible to consider the more general problem, supposing only that

$$c_1\sigma^2 < \sigma f(\sigma) < c_2\sigma^2 \tag{13}$$

We construct $n(n + 1)/2$ numbers

$$r_{ik} \ (i, k = 1, 2, \ldots, n), \qquad r_{ik} = r_{ki} \tag{14}$$

by means of the formulas

$$r_{i\alpha} = \sum_{k=1}^{n} m_{ki} b_{k\alpha} + \sum_{k=1}^{n} m_{k\alpha} b_{ki} + c\left(j_\alpha \sum_{s=1}^{n} m_{si} h_s + j_i \sum_{s=1}^{n} m_{s\alpha} h_s\right) + + cp\left(j_\alpha \sum_{s=1}^{n} j_s b_{si} + j_i \sum_{s=1}^{n} j_s b_{s\alpha}\right) + c^2 j_i j_\alpha \left(\sum_{s=1}^{n} j_s h_s\right) p \tag{15}$$

where

$$m_{ik} = m_{ki} \qquad (i, k = 1, 2, \ldots, n) \tag{16}$$

are elements of a certain symmetric matrix, and p is a real number.

The following theorem holds.

Theorem. For the stability of system (1) with any function (13) it is sufficient:

1. That the roots of the equation

$$D(\lambda) - (c_1 + \varepsilon) M(\lambda) = 0, \qquad D(\lambda) - (c_2 - \varepsilon) M(\lambda) = 0$$

for all sufficiently small $\epsilon > 0$ be roots with negative real parts;

2. That there exist real numbers m_{ka} and p, for which the quadratic form

$$U = \sum\sum r_{s\alpha} x_s x_\alpha$$

is of definite sign for all

$$c_1 < c < c_2 \tag{17}$$

It is obvious, that condition 1 of the theorem turns out also to be necessary for stability for arbitrary functions (13).

Condition 2 is equivalent to the inequalities

$$\Delta_k(c) > 0 \qquad (k = 1, \ldots, n), \qquad c_1 < c < c_2 \tag{18}$$

where $\Delta_k(c)$ is Sylvester's determinant of order k composed of the numbers r_{sk}.

Proof. Let the inequalities (18) be satisfied by some m_{ia}, β. Then by virtue of condition 1 and Liapunov's theorem the quadratic form

$$V_k = \frac{1}{2} \sum\sum m_{i\alpha} x_i x_\alpha + p \frac{c\sigma^2}{2} = \frac{1}{2} \sum\sum (m_{i\alpha} + pcj_i j_\alpha) x_i x_\alpha \tag{19}$$

is strictly negative for $c = c_1 + \epsilon$ and $c = c_2 - \epsilon$, where ϵ is an arbitrary small positive number, since

$$\frac{dV_k}{dt} = U$$

From this it follows that the quadratic form V_k is strictly negative for all c satisfying condition (17).

Consequently, the linear system (7) under condition (17) has a Liapunov function of the form (6), but then it is possible to verify [8] that the function

$$V = \frac{1}{2} \sum\sum m_{i\alpha} x_i x_\alpha + \beta \int_0^\sigma f(\sigma)\, d\sigma$$

is a Liapunov function for system (1) under conditions (13), as required.

The mentioned conditions of stability are the broadest that can be obtained by using Liapunov functions of the form (3), having a derivative of definite sign by virtue of system (1). The arguments of the present note are easily extended to systems with several control units.

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